

An algebraic characterization of simple closed curves on surfaces with boundary

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Abstract We characterize in terms of the Goldman Lie algebra which conjugacy classes in the fundamental group of a surface with non empty boundary are represented by simple closed curves.

We prove the following: *A non power conjugacy class X contains an embedded representative if and only if the Goldman Lie bracket of X with the third power of X is zero.*

The proof uses combinatorial group theory and Chas' combinatorial description of the bracket recast here in terms of an exposition of the Cohen-Lustig algorithm. Using results of Ivanov, Korkmaz and Luo there are corollaries characterizing which permutations of conjugacy classes are related to diffeomorphisms of the surfaces.

This result completes the solution of a problem posed by Turaev in the eighties.

Our main theorem counts the minimal possible number of self-intersection points of representatives of a conjugacy class X in terms of the bracket of X with the third power of X .

1 Introduction

During the middle eighties Goldman [10] defined a Lie algebra structure on the \mathbb{Z} -module generated by non-trivial conjugacy classes and explained it as a universal Poisson structure fitting with calculations of Scott Wolpert [28] using the symplectic structure on Teichmüller space. In the late eighties Turaev added a Lie cobracket structure [27] on this \mathbb{Z} -module and showed the two operations, bracket and cobracket, defined a Lie bialgebra in the sense of Drinfeld. Turaev formulated a question whether the vanishing of the cobracket characterized embedded non-power conjugacy classes. Turaev was motivated by the group theory statement:

Every surjection $\pi_g \xrightarrow{s} F_g \times F_g$ contains in its kernel an embedded conjugacy class.

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Here π_g is the fundamental group of a closed surface of genus g , F_g is the free group on g generators and the surjection is the data of a Heegaard decomposition of a simply connected three manifold. This statement was shown to be equivalent to the Poincaré conjecture ([16] and [26].)

Counterexamples to Turaev's conjecture were given in [5] and evidence was obtained for a bracket characterization of embedded conjugacy classes. Before the counterexamples were found, attempts to answer Turaev's question led to the String Topology paper [3].

In this paper we also report on the empirical evidence that the vanishing of the cobracket of non-trivial powers of a conjugacy class may also characterize embedded classes.

The main theorem of this paper, which holds for surfaces with non-empty boundary, is:

Theorem 4.1: *Let \mathcal{V} be a non-power conjugacy class of the fundamental group of an oriented surface. Let p and q be distinct positive integers with one of them, p or q at least three. The number of terms (counted with multiplicity) of the Goldman bracket $\langle \mathcal{V}^p, \mathcal{V}^q \rangle$ is $2 \cdot p \cdot q$ times the minimal possible number of self-intersections of representatives of the conjugacy class \mathcal{V} .*

Our proof of the main theorem really happens in the analysis of Proposition 2.19. This result distinguishes the conjugacy classes of various sets of linear words. The analog of this proposition for closed surfaces is work in progress and may yield an analogous characterization of embedded conjugacy classes there. Beyond Proposition 2.19, the proof uses the combinatorial presentation of the Lie bialgebra from [5]. However, for the purposes of a simpler and more self-contained exposition we adapt propositions of Cohen-Lustig [7] to arrive in this paper at a modified combinatorial presentation of the Goldman Lie algebra.

The surjection above can be realized by a simplicial map from the surface to the Cartesian square of a trivalent graph. Discovering some structure analogous to the Goldman bracket for the latter space and obtaining a deeper understanding of the comitant algebra presented here could possibly lead to a topological or algebraic proof of the group theoretic statement above which is now known to be true by Perelman's work ([21], [22] and [23]). See also Morgan-Tian [13]).

The paper closes with some further algebraic questions and problems which are related to the ideas here.

In more detail, the paper is organized as follows: first we develop the combinatorial aspects of our work (Section 2) and then, the topologic-geometric aspects (Section 3). More precisely, in Section 2 we define linear and cyclic words, an equivalence relation on the set of cyclic permutations of pairs of words and an order on the set of half infinite words. These concepts were already introduced in [2] and [7]. In this paper, for

completeness, we give proofs of all the combinatorial results we state. Using all these elements, we define a certain bilinear map, the combinatorial Goldman Lie bracket. Also, we prove that certain pairs of cyclic words cannot be conjugate (Proposition 2.19). The argument of Proposition 2.19 (especially Figure 3) is the heart of the proof of the main result of this paper.

In Section 3, we define boundary expansions and we show that the Goldman bracket is the bilinear map we defined combinatorially in Section 2.

Even if for Section 2 the definitions and results of Section 3 are not needed, we will make reference to Section 3 when exposing Section 2 in order to clarify and give meaning to an otherwise challenging combinatorial discussion.

Finally, in Section 4, we prove our main result, Theorem 4.1, saying that the bracket "counts" the number of self-intersection points of a conjugacy class. We conclude with Section 5 by stating some questions and conjectures relative to these problems.

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2 Cyclic words, linking pairs and a bilinear map

In this section we introduce the free \mathbb{Z} -module \mathbb{V} of cyclic words of certain alphabet and define a bilinear map $[\cdot, \cdot]: \mathbb{V} \otimes \mathbb{V} \longrightarrow \mathbb{V}$. From the definition of $[\cdot, \cdot]$, it will follow straightforwardly that $[\cdot, \cdot]$ is antisymmetric. In fact, this bilinear map satisfies the Jacobi identity. We will give an indirect proof of this fact, by showing that the combinatorial bilinear map and the Goldman Lie bracket of curves on a surface coincide. In [11] a purely combinatorial proof of the Jacobi identity will be exhibited.

The bilinear map $[\cdot, \cdot]$ coincides with the one defined in [5], but the presentation we give here, is easier to process for the reader familiar with Hyperbolic Geometry.

2.1 Linear and cyclic words

In this subsection we introduce definitions and recall basic well known results about linear and cyclic words.

Let q be a positive integer. A q -*alphabet* or, briefly, an *alphabet* is a set of $2q$ symbols, $\{a_1, a_2, \dots, a_q, \bar{a}_1, \bar{a}_2, \dots, \bar{a}_q\}$, endowed with a fixed linear order. We denote a q -alphabet by \mathbb{A}_q . (There are approximately $\binom{2q}{q}$ different discussions depending on the order of \mathbb{A}_q .) The elements of \mathbb{A}_q are *letters*. A *linear word in \mathbb{A}_q* is a non-empty finite

sequence of symbols $v_0v_1 \dots v_{n-1}$ such that v_i belongs to \mathbb{A}_q for each $i \in \{0, 1, \dots, n-1\}$. For each letter v , $\overline{v} = v$.

Let $V = v_0v_1 \dots v_{n-1}$ be a linear word. We say that n , the number of letters of V is the *length* of V . By definition, $\overline{V} = \overline{v}_{n-1}\overline{v}_{n-2} \dots \overline{v}_0$. The linear word V is *freely reduced* if $v_i \neq \overline{v}_{i+1}$ for each $i \in \{0, 1, \dots, n-1\}$. If V is freely reduced and $v_{n-1} \neq \overline{v}_0$ then V is *cyclically reduced*.

Notation 2.1. When dealing with letters denoting linear words $V = v_0v_1 \dots v_{n-1}$ we will always consider subindices of letters mod the length of V , that is n . Thus if $V = v_0v_1 \dots v_{n-1}$ is a linear word and i is an integer, v_i denotes the letter v_h of V where h is the only non-negative integer such that n divides $i - h$ and $h < n$.

Consider the equivalence relation on the set of linear words, generated by the pairs of the form (V, W) such that V is a cyclic permutation of W or $V = Wv\overline{v}$ where v is a letter in \mathbb{A}_q . The equivalence classes under this equivalence relation are called *cyclic words*. (Observe that these are the conjugacy classes of the free group generated by a_1, a_2, \dots, a_q). Thus, every cyclic word can be labeled by a unique reduced oriented ring of symbols. (see Figure 1). If V is a (not necessarily reduced) linear word, we denote the equivalence class of V by \widehat{V} . Observe that the definition of cyclic word we are giving here does not coincide with the one given in [5] which allowed unreduced rings of symbols.

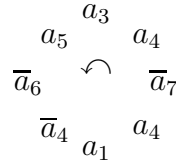


Figure 1: A ring in the letters of \mathbb{A}_7

If $V = v_0v_1 \dots v_{n-1}$ is a linear word and j is an integer then the linear word $v_jv_{j+1} \dots v_{n-1}v_0 \dots v_{j-1}$ is denoted by V_j . Notice that the length of V equals the length of V_j .

A linear word V is a *linear representative* of a cyclic word \mathcal{V} if V is cyclically reduced and belongs to the equivalence class \mathcal{V} . Observe that all the linear representatives of \mathcal{V} are exactly all the cyclic permutations of V . Thus, in particular, all the linear representatives of a cyclic word \mathcal{V} have the same length.

If \mathcal{V} is a cyclic word, V a linear representative and k an integer, we define \mathcal{V}^k as $\widehat{V^k}$ if k is positive and as $\widehat{(\overline{V})^{-k}}$ if k is negative. (These are the basic operations on conjugacy classes in groups and are well defined by these prescriptions). A linear (resp. cyclic) word is *primitive* if it cannot be written as V^r (resp. \mathcal{V}^r) for some $r \geq 2$ and some reduced linear word V (resp. cyclic word \mathcal{V}). The *length* of a cyclic word is the length of any linear representative. (Recall that, by definition, a linear representative of a cyclic word is cyclically reduced.)

We start by gathering together some elementary well known results we will need throughout these pages.

- Lemma 2.2.** (1) *If V is a linear word and $V = V_i$ for some integer i then either V is not a primitive word or i is a multiple of the length of V .*
- (2) *If V is a cyclically reduced linear word then there exists a primitive linear word W and a positive integer k such that $V = W^k$.*
- (3) *If V and W are cyclically reduced words such that $\widehat{V} = \widehat{W}$ then there exists an integer i such that $V = W_i$.*
- (4) *If V is a cyclically reduced linear word and k and l are positive integers then $(V_k)^l = (V^l)_k$.*

2.2 An equivalence relation on the set of cyclic permutations of pairs of words

Let V and W be two cyclically reduced words of length n and m respectively. In this Subsection we define an equivalence relation on the set of ordered pairs in $\{0, 1, \dots, n-1\} \times \{0, 1, \dots, m-1\}$. This definition is purely combinatorial but has the following geometric interpretation: This equivalence relation is such that there exists a one to one correspondence between pairs of integers equivalent to (i, j) and fundamental domains traversed simultaneously by the two axes of the two hyperbolic transformations associated in Subsection 3.1 to the words V and W and the pair (i, j) . More precisely, in the context of Subsection 3.1, given two cyclically reduced words V and W and (i, j) , one can determine which fundamental domains are traversed by V_i and W_j "starting" at D_1 . There is a natural bijection between the elements of $\widetilde{(i, j)}$ and the pair of fundamental domains that are traversed by the axis of V_i and the axis of W_j .

Definition 2.3. Let $V = v_0v_1\dots v_{n-1}$ and $W = w_0w_1w_2\dots w_{m-1}$ be two cyclically reduced words in \mathbb{A}_q . Denote by $I(V, W)$ the set of ordered pairs of integers $\{(j, k) : 0 \leq j < n \text{ and } 0 \leq k < m\}$ and by $\mathcal{R}(V, W)$ the equivalence relation on $I(V, W)$ generated by the pairs

- (1) $(j, k) \sim (j+1, k+1)$ if $v_j = w_k$.
- (2) $(j+1, k) \sim (j, k+1)$ if $v_j = \overline{w}_k$.

Denote by $\widetilde{(i, j)}$ the equivalence class of (i, j) . □

We use with the entries of pairs in $I(V, W)$ the same convention we use with subindices in words: namely, integers are taken mod the length of the corresponding word.

Next we prove an auxiliary result which will be used in the proof of Proposition 2.5.

Lemma 2.4. *Let \mathbf{V} and \mathbf{W} be two cyclically reduced words. Let $(i, j), (k, h)$ and (l, f) be elements of $\mathbf{I}(\mathbf{V}, \mathbf{W})$. If (i, j) and (k, h) are related as in Definition 2.3(1) then (k, h) and (l, f) are not related as in Definition 2.3(2).*

Proof. Since (i, j) and (k, h) are related as in Definition 2.3(1) there are two possibilities.

- (a) $(k, h) = (i + 1, j + 1)$ and $v_i = w_j$
- (b) $(k, h) = (i - 1, j - 1)$ and $v_{i-1} = w_{j-1}$

Suppose that (k, h) and (l, f) are related as in Definition 2.3(2). Then there are two possibilities.

- (i) $(l, f) = (k + 1, h - 1)$ and $v_k = \overline{w}_{h-1}$
- (ii) $(l, f) = (k - 1, h + 1)$ and $v_{k-1} = \overline{w}_h$

Assume that (a) and (i) hold. Then $v_i = w_j = w_{h-1} = \overline{v}_k = \overline{v}_{i+1}$. Then \mathbf{V} is not reduced, a contradiction. The other three cases follow by similar arguments. ■

In the next proposition we give a characterization of the equivalence classes of $\mathcal{R}(\mathbf{V}, \mathbf{W})$.

Proposition 2.5. *Let \mathbf{V} and \mathbf{W} be two cyclically reduced words. If C is an equivalence class of $\mathcal{R}(\mathbf{V}, \mathbf{W})$ then there exists an ordered pair of integers (i, j) such that exactly one of the following holds.*

- (1) $C = \{(i, j), (i + 1, j + 1), \dots, (i + c, j + c)\}$ for some non-negative integer c . Moreover $v_{i+c} \neq w_{j+c}$.
- (2) $C = \{(i, j), (i + 1, j - 1), \dots, (i + c, j - c)\}$ for some positive integer c . Moreover $v_{i+c} \neq \overline{w}_{j-c-1}$.
- (3) $v_{i+s} = w_{j+s}$ for every integer s .
- (4) $v_{i+s} = \overline{w}_{j-s}$ for every integer s .

Moreover, in cases (1) and (2), if the length of \mathbf{V} and the length of \mathbf{W} are equal to n for some positive integer n then $c < n$.

Proof. If C contains exactly one element, then C is of type (1) and the results holds. Thus we can assume now that C contains more than one element. Then C contains two elements related as in Definition 2.3(1) or two elements related as in Definition 2.3(2).

We assume that C contains two elements related as in Definition 2.3(1). The other case can be studied in an analogous way.

Let (h, k) be an element in C . By Lemma 2.4, there are no pairs of elements in C related as in Definition 2.3(2). Denote by T the set of integers such that $(h + t, k + t)$ belongs to C . Note that every element of C has the form $(h + t, k + t)$ for some integer t in T . By definition, zero belongs to T so T is a non-empty set. It is not hard to see that T has an upper bound if and only if T has a lower bound. Assume that T has an upper bound. Let u (resp. l) denote the maximum (res. minimum) of T . Set $i = h - l$ and $j = k - l$. Then $C = \{(i, j), (i + 1, j + 1), \dots, (i + u + l, j + u + l)\}$ $(i + u + l + 1, j + u + l + 1) = (h + u + 1, k + u + 1) \notin C$. This implies that $v_{h+u} \neq w_{h+u}$. Thus C is of type (1).

On the other hand, if T does not have an upper bound then C is of type (3).

Assume now that the length of V and the length of W are equal to n for some positive integer n . Observe that if T contains at least n integers, then T contains all the integers. This completes the proof. ■

In the conditions of Proposition 2.5, to each equivalence class C we associate a non-negative integer which will be called the *negative length of C* . The negative length is defined as the number c if (2) holds and 0 otherwise.

Let V and W be two cyclically reduced linear words. In Subsection 2.4, in order to define the bilinear map, we need to choose a representative (i, j) of an equivalence class of $\mathcal{R}(V, W)$ and consider $\widehat{V_i W_j}$. By Lemma 2.7, this is independent of the representative (i, j) . Nevertheless, in order to facilitate the proof of Theorem 3.16, we will use a representative with certain properties, which is defined as follows. A pair of integers (i, j) in $I(V, W)$ is said to be *extremal* if $v_i \neq w_j$ and $v_i \neq \overline{w}_{j-1}$.

Corollary 2.6. *Let V and W be two cyclically reduced linear words and let C be an equivalence class of $\mathcal{R}(V, W)$ as in Lemma 2.5(1) or (2). Then C contains a unique extremal pair.*

The following lemma follows straightforwardly from Proposition 2.5.

Lemma 2.7. *Let V and W be two cyclically reduced linear words. Let A be a map from $I(V, W)$ to the set of cyclic words in \mathbb{A}_q defined by $A(i, j) = \widehat{V_i W_j}$. Then A is constant over each equivalence class of $\mathcal{R}(V, W)$.*

Moreover, if c is the negative length of (i, j) , and n and m are, respectively, the lengths of V and W then the length of $\widehat{V_i W_j}$ is $n + m - 2c$.

Example 2.8. Consider the ordered alphabet $\mathbb{A}_2 = \{a, b, \overline{a}, \overline{b}\}$. For each of the words V and W below, we list the set of equivalence classes of $\mathcal{R}(V, W)$. The pairs in bold

are the extremal pairs (note that certain equivalence classes do not contain extremal pairs).

- (1) $V = ab$, $W = aab$, $\{(0, 0), (\mathbf{1}, \mathbf{1}), (0, 1), (1, 2)\}, \{(\mathbf{1}, \mathbf{0})\}, \{(0, 2)\}$
- (2) $V = W = aabb$, $\{(0, 0), (1, 1), (2, 2), (3, 3)\}, \{(0, 1), (\mathbf{1}, \mathbf{2})\}, \{(1, 0), (\mathbf{2}, \mathbf{1})\}, \{(2, 3), (\mathbf{3}, \mathbf{0})\}, \{(3, 2), (\mathbf{0}, \mathbf{3})\}, \{(\mathbf{0}, \mathbf{2})\}, \{(\mathbf{2}, \mathbf{0})\}, \{(\mathbf{1}, \mathbf{3})\}, \{(\mathbf{3}, \mathbf{1})\}$
- (3) $V = aab$, $W = \bar{a}$, $\{(0, 0), (1, 0), (\mathbf{2}, \mathbf{0})\}$.

□

2.3 Linking pairs and sign

The set of reduced half infinite words in the ordered alphabet \mathbb{A}_q is in correspondence with a subset of points in the boundary of the Poincaré disk \mathbb{D} (see Lemma 3.5). In this Subsection, we will study following [2] an order on the set of half infinite words in \mathbb{A}_q , which reflects the cyclic order of the corresponding points in the boundary of \mathbb{D} . Using this order, for every pair of words V and W , we associate an integer, -1 , 1 or 0 to each class in $\mathcal{R}(V, W)$. We will see in Subsection 3.3 that this combinatorial definition is related to whether two axis of certain hyperbolic transformation intersect and if they do intersect, which is the sign of that intersection.

We now define an order in the set of half infinite words in \mathbb{A}_q . Let $T = t_0 t_1 t_2 \dots$ and $U = u_0 u_1 u_2 \dots$ be two distinct half infinite words. We say that $T < U$ if one of the following holds.

- (i) $t_0 < u_0$
- (ii) there exists a non-negative integer j such that $t_i = u_i$ for every i such that $0 \leq i \leq j$ and t_{j+1} comes before u_{j+1} in the new alphabet obtained by cyclically permuting the order of \mathbb{A}_q in such a way that \bar{t}_j is the first element.

Note that for every pair of words T and U , either $T < U$ or $U < T$ or $T = U$. In other words, the order is linear.

A finite sequence of half infinite words T_1, T_2, \dots, T_k is *linearly ordered* if $T_1 < T_2 < \dots < T_k$ or $T_k < \dots < T_2 < T_1$. The sequence T_1, T_2, \dots, T_k is *cyclically ordered* if a cyclic permutation of T_1, T_2, \dots, T_k is linearly ordered.

Remark 2.9. The alphabet \mathbb{A}_q is initially endowed with a linear order. Any cyclic permutation of this linear order yields the same cyclic order on the set of infinite words on the letters of \mathbb{A}_q , and this cyclic order is what determines all our further results. The reason why one initially endows \mathbb{A}_q with a linear order and not with a cyclic order is that linear orders are easier to handle. □

Given a finite linear word $V = v_0v_1v_2 \dots v_{n-1}$ we define the infinite word V^∞ as $VV \dots$ and $V^{-\infty}$ and $\overline{V}\overline{V} \dots$.

Example 2.10. Consider the ordered alphabet $\mathbb{A}_2 = \{a, b, \bar{a}, \bar{b}\}$ and $V = aabb$. In the above order we have

$$V_0^\infty < V_1^\infty < V_3^\infty < V_2^\infty < \overline{V}_2^\infty < \overline{V}_1^\infty < \overline{V}_3^\infty < \overline{V}_0^\infty$$

□

Let $V = v_0v_1v_2 \dots v_{n-1}$ and $W = w_0w_1w_2 \dots w_{m-1}$ be two cyclically reduced words in \mathbb{A}_q . To each ordered pair of integers $(i, j) \in I(V, W)$ we associate an element of $\{1, -1, 0\}$ as follows

$$s_{V,W}(i, j) = \begin{cases} 1 & \text{if } V_i^\infty, W_j^\infty, V_i^{-\infty}, W_j^{-\infty} \text{ is cyclically ordered} \\ -1 & \text{if } V_i^\infty, W_j^{-\infty}, V_i^{-\infty}, W_j^\infty \text{ is cyclically ordered} \\ 0 & \text{otherwise.} \end{cases}$$

A proof of a variant of the following lemma, using hyperbolic geometry can be found in [7]. Here, for completeness, we give the argument.

Lemma 2.11. *Let V and W be two cyclically reduced linear words. The function $s_{V,W}$ is constant on each equivalence class of $\mathcal{R}(V, W)$.*

Proof. It is enough to prove that if two pairs are related as in Definition 2.3(1) or (2) then $s_{V,W}$ has the same value for these two pairs. We study the first case. The second follows similarly. Consider two pairs (i, j) and $(i+1, j+1)$ in $I(V, W)$ for which definition Definition 2.3(1) holds. Thus $v_i = w_j$. Assume that

$$V_i^\infty < W_j^\infty < \overline{V}_i^\infty < \overline{W}_j^\infty.$$

(The other possibilities follows by analogous arguments). Since V is cyclically reduced, $v_i \neq \bar{v}_{i-1}$. By definition of order we have, $w_j = v_i < \bar{v}_{i-1} \leq \bar{w}_{j-1}$.

Assume first that $v_{i+1} = w_{j+1}$. Since $V_i^\infty < W_j^\infty$ and $V_i^\infty = v_iv_{i+1}V_{i+2}^\infty$ and $W_j^\infty = v_iv_{i+1}W_{j+1}^\infty$, $V_{i+1}^\infty < W_{j+1}^\infty$. We consider the following two cases

- (1) $v_{i-1} = w_{j-1}$. Since $\overline{V}_i^\infty < \overline{W}_j^\infty$, and $\overline{V}_i^\infty, \overline{W}_j^\infty$ both start with the same letter, $\overline{V}_{i+1}^\infty = \bar{v}_i\bar{v}_{i-1}\overline{V}_{i-1}^\infty < \bar{v}_i\bar{v}_{i-1}\overline{W}_{i-1}^\infty = \overline{W}_{i+1}^\infty$. The words V_{i+1}^∞ and W_{j+1}^∞ (resp. $\overline{V}_{i+1}^\infty$ and $\overline{W}_{j+1}^\infty$) start with the same letter. Then the cyclic order of the sequences $V_i^\infty, W_j^\infty, \overline{V}_i^\infty, \overline{W}_j^\infty$ and $V_{i+1}^\infty, W_{j+1}^\infty, \overline{V}_{i+1}^\infty, \overline{W}_{j+1}^\infty$ coincide.
- (2) $v_{i-1} \neq w_{j-1}$. Hence, $v_i = w_j < \bar{v}_{i-1} < \bar{w}_{j-1}$. This inequality implies that

$$\overline{V}_{i+1}^\infty = \bar{v}_i\bar{v}_{i-1}\overline{V}_{i-1}^\infty < \bar{w}_j\bar{w}_{j-1}\overline{W}_{j-1}^\infty.$$

Since $V_{i+1}^\infty < W_{j+1}^\infty$ and both words V_{i+1}^∞ and W_{j+1}^∞ start with the same letter, this case is done.

Finally assume that $v_{i+1} \neq w_{j+1}$. Then v_{i+1} comes before w_{j+1} in the alphabet obtained by cyclically permuting \mathbb{A}_q so that \bar{v}_i is the first element. Since V_{i+1}^∞ starts with v_{i+1} , W_{j+1}^∞ starts with w_{j+1} and both words \bar{V}_{i+1}^∞ and \bar{W}_{j+1}^∞ start with \bar{v}_i , if we can show that $\bar{V}_{i+1}^\infty < \bar{W}_{j+1}^\infty$, the result follows. We consider the following cases

- (1) $v_{i-1} = w_{j-1}$. As in (1) of the previous case, $\bar{V}_{i+1}^\infty < \bar{W}_{j+1}^\infty$.
- (2) $v_{i-1} \neq w_{j-1}$. Hence, $v_i = w_j < \bar{v}_{i-1} < \bar{w}_{j-1}$. Therefore,

$$\bar{v}_i \bar{v}_{i-1} \bar{V}_{i-1}^\infty = \bar{V}_{i+1}^\infty < \bar{W}_{j+1}^\infty = \bar{w}_j \bar{w}_{j-1} \bar{W}_{j-1}^\infty.$$

■

Example 2.12. If V and W be as in Example 2.8(1) then the class that contains $(0, 0)$ has sign -1 . The other two classes have sign zero.

If $V = W = aabb$ as in Example 2.8(2) by the calculations in Example 2.10 one can deduce the following: $s_{v,v}(1, 3) = 1$, $s_{v,v}(3, 1) = -1$ and the sign of the classes that do not contain $(1, 3)$ or $(3, 1)$ is zero.

Finally, if V and W are as in Example 2.8(3), the sign of the unique equivalence class is one. □

Definition 2.13. Let V and W be two cyclically reduced linear words. An ordered pair of integers (i, j) in $I(V, W)$ such that $s_{v,w}(i, j) \neq 0$ is a *linking pair*. The set of equivalence classes $\widetilde{(i, j)}$ such that (i, j) is a linking pair is denoted by $\mathcal{LP}(V, W)$. Observe that $\mathcal{LP}(V, W)$ is well defined because of Lemma 2.11. □

Lemma 2.14. Let V and W be two cyclically reduced linear words and let C be a class in $\mathcal{R}(V, W)$. If C is as in Lemma 2.5(3) or (4) then C is not a linking pair. In symbols, $C \notin \mathcal{LP}(V, W)$.

Proof. If C is as in Lemma 2.5(3) or (4) then $V_i^\infty = W_j^\infty$ or $V_i^\infty = \bar{W}_j^\infty$ for each $(i, j) \in C$. Thus, by definition $s_{v,w}(i, j) = 0$. ■

Remark 2.15. For each pair of cyclically reduced words V and W , it is not hard to prove that there is a one to one correspondence between the linking pairs $\mathcal{LP}(V, W)$ introduced in [7] and the linked pairs of [5]. □

2.4 A bilinear map on free \mathbb{Z} -module of cyclic words

Let \mathbb{V} denote the free \mathbb{Z} -module generated by the set of cyclic words in \mathbb{A}_q .

Remark 2.16. By definition, the cyclic word consisting of the empty word is an element of the base of \mathbb{V} . Had we chosen to take \mathbb{V} as the free \mathbb{Z} -module generated

by all the cyclic words with the exception of the cyclic word consisting of the empty word then all the results of this combinatorial section are still valid. The topological results about the Lie algebra still hold, but of course we will be studying non-trivial free homotopy classes of curves on a surface. \square

We will define a bilinear map on $[\cdot, \cdot]: \mathbb{V} \otimes \mathbb{V} \longrightarrow \mathbb{V}$. To do so, it is enough to define it in each ordered pair of elements of the base and extend it by linearity. By Lemma 2.2(2) every cyclic word can be written as $\widehat{\mathbf{V}}^k$ for some positive integer k where \mathbf{V} is a primitive cyclically reduced linear word.

Let \mathbf{V} and \mathbf{W} be two primitive cyclically reduced linear words and let k and l be two positive integers. Set

$$[\widehat{\mathbf{V}}^k, \widehat{\mathbf{W}}^l] = \sum_{\widetilde{(i,j)} \in \mathcal{LP}(\mathbf{V}, \mathbf{W})} k \cdot l \cdot s_{\mathbf{V}, \mathbf{W}}(i, j) \cdot \widehat{\mathbf{V}}_i^k \widehat{\mathbf{W}}_j^l$$

We claim that the bilinear map $[\cdot, \cdot]$ is well defined. Indeed, by Lemmas 2.11 and 2.7, the product

$$s_{\mathbf{V}, \mathbf{W}}(i, j) \cdot \widehat{\mathbf{V}}_i^k \widehat{\mathbf{W}}_j^l$$

does not depend on the representative of $\widetilde{(i, j)}$. On the other hand, it is easy to check the definition of the bilinear map $[\cdot, \cdot]$ does not depend on the choice of the linear representatives of the two cyclic words.

Example 2.17. Let \mathbf{V} and \mathbf{W} be as in Example 2.8(1) then $[\widehat{\mathbf{V}}, \widehat{\mathbf{W}}] = -\widehat{abaaab}$.

If $\mathbf{V} = \mathbf{W} = aabb$ as in Example 2.8(2),

$$[\widehat{\mathbf{V}}, \widehat{\mathbf{V}}^3] = 3abbabaabbaabbaab - 3baababbaabbaabba.$$

(Observe that this is an example where the Turaev cobracket of $\widehat{\mathbf{V}}$ is zero but \mathbf{V} does not have a simple representative, see Remark 3.17 and [5])

Finally, if \mathbf{V} and \mathbf{W} are as in Example 2.8(3), then $[\widehat{\mathbf{V}}, \widehat{\mathbf{W}}] = \widehat{ab}$. \square

Remark 2.18. It is not hard to prove that for any pair of cyclically reduced words, \mathbf{X} and \mathbf{Y} , not necessarily primitive, $[\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}] = \sum_{\widetilde{(i,j)} \in \mathcal{LP}(\mathbf{X}, \mathbf{Y})} s_{\mathbf{X}, \mathbf{Y}}(i, j) \cdot \widehat{\mathbf{X}}_i \widehat{\mathbf{Y}}_j$. Thus we could have used this formula as a definition of the bilinear map $[\cdot, \cdot]$. We have chosen the other formula because it makes our proofs shorter. \square

2.5 Certain words are not conjugate

Let \mathbf{X} be a cyclically reduced primitive word. In this subsection we prove that certain pairs of cyclic words we construct out of \mathbf{X} , cannot be conjugate. The conjugacy classes of each of the terms of the bracket $[\widehat{\mathbf{X}}^p, \widehat{\mathbf{X}}^q]$ we will study later on have the same form

as these cyclic words. (Recall that for each linear word Y , \widehat{Y} is the conjugacy class of Y .)

Here is an outline of the proof: if $X_i^p X_j^q$ and $X_k^p X_h^q$ are conjugate then a subword P of X_i^p or X_j^q is a (cyclic) subword of $X_k^p X_h^q$ (Observe that we cannot assume P is equal to X_i^p or X_j^q because there might be cancellation). When the length of P is larger than the length of X , certain letters repeat in P and so in $X_k^p X_h^q$. Some of these repetitions will contradict the fact that (i, j) and (h, k) are linked pairs (see Figure 3).

In order to prove Proposition 2.19 for $p + 2 \leq q$ one only needs to consider the first three cases studied in the proof (Case 1, 2 and 3). Case 4, 5 and 6 are needed for the case $p + 1 = q$.

Using the notation of Definition 2.3 we have,

Proposition 2.19. *Let X be a primitive cyclically reduced linear word and let (i, j) and (k, h) be linking pairs which are not equivalent under $\mathcal{R}(X, X)$. Let p and q be distinct positive integers such that $p \geq 3$ or $q \geq 3$. Then $\widehat{X_i^p X_j^q} \neq \widehat{X_k^p X_h^q}$.*

Proof. We can assume without loss of generality that $p < q$. Thus, $q > 3$.

Since (i, j) and (k, h) are linking pairs, by Lemma 2.14, the classes of (i, j) and (k, h) are as in Proposition 2.5(1) or (2). By Corollary 2.6 and Lemma 2.7 we can assume that (i, j) and (k, h) are extremal. Since (i, j) and (k, h) are not equivalent, in particular, $(i, j) \neq (k, h)$.

We argue by contradiction. If $\widehat{X_i^p X_j^q} = \widehat{X_k^p X_h^q}$ by Lemma 2.7, the negative length of the class (i, j) and the negative length of the class (k, h) are equal. Denote the common negative length by c . By Lemma 2.7, the length of the cyclic word $\widehat{X_i^p X_j^q}$ is $(p + q)l - 2c$, where l denotes the length of X . Moreover, by Proposition 2.5 $c < l$.

Let $V = v_0 v_1 \dots v_{(q+p)l-2c-1}$ and $W = w_0 w_1 \dots w_{(q+p)l-2c-1}$ be the linear words defined by the formulae

$$v_m = \begin{cases} x_{j+m+c} & \text{if } 0 \leq m < ql - c \\ x_{i+m+c} & \text{if } ql - c \leq m < (p + q)l - 2c \end{cases}$$

and

$$w_m = \begin{cases} x_{h+m+c} & \text{if } 0 \leq m < ql - c \\ x_{k+m+c} & \text{if } ql - c \leq m < (p + q)l - 2c \end{cases}$$

Let $a, b \in \{0, 1, 2, \dots, (p + q)l - 2c\}$. By $X_{a,b}$ we denote the subword of \widehat{X} defined by $x_a x_{a+1} \dots x_{b-1}$. Using this notation we can represent the word W as in Figure 2.

Since (i, j) and (h, k) are extremal, $x_i \neq \overline{x}_{j-1}$ and $x_h \neq \overline{x}_{k-1}$. By hypothesis, X is cyclically reduced. Thus V and W are reduced. Since V and W have length $(q + p)l - 2c$ both words are cyclically reduced. Therefore, V is a linear representative of $\widehat{X_i^p X_j^q}$ and

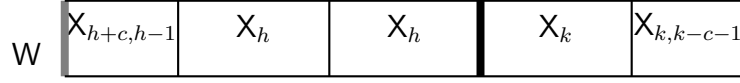


Figure 2: Graphic representation of W .

W is a linear representative of $\widehat{X_k^p X_h^q}$. Since $\widehat{X_i^p X_j^q} = \widehat{X_k^p X_h^q}$ by Lemma 2.2(3) there exists a positive integer r , such that $V = W_r$. Hence,

$$v_m = w_{m+r} \text{ for every integer } m \quad (1)$$

We can assume without loss of generality that $0 \leq r < (q+1)l - 2c$. By the definition of V and W , we have

$$v_m = v_{m+l} \text{ if } 0 \leq m < (q-1)l - c \quad (2)$$

$$w_m = w_{m+l} \text{ if } 0 \leq m < (q-1)l - c \quad (3)$$

We will complete the proof by showing that all possible values of r and c lead to a contradiction.

Case 1: $r = 0$. Here $V = W$ and so $X_i = X_k$ and $X_j = X_h$. Since (i, j) and (k, h) are distinct pairs then $i \neq k$ or $j \neq h$. By Lemma 2.2(1), X is not a primitive word contradicting our assumptions.

Case 2: $0 < r \leq (q-1)l - c$. Thus $0 \leq (q-1)l - c - r < (q-1)l - c$. By Equation (2), $v_{(q-1)l-c-r} = v_{ql-c-r}$. Hence, by Equation (1)

$$x_k = w_{ql-c} = v_{ql-c-r} = v_{(q-1)l-c-r} = w_{(q-1)l-c} = x_h,$$

contradicting the hypothesis that (h, k) is extremal (see Figure 3).

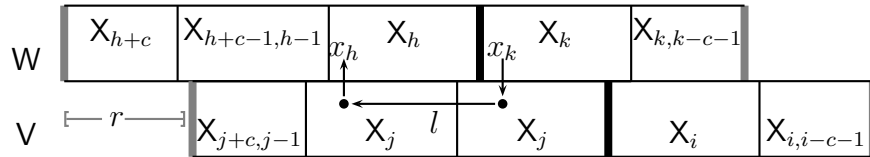


Figure 3: Case $0 < r \leq (q-1)l - c$.

Case 3: $(p+1)l - c < r < (p+q)l - 2c$. Set $r' = (p+q)l - 2c - r$. Thus, $0 < r' < (q-1)l - c$ and $W_m = V_{m-r} = V_{m+r'}$. Then we can proceed as in Case 2 (with r' instead of r and the roles of V and W interchanged) to complete the proof.

If $p+2 \leq q$ then the proof is complete. Now, we need to complete the proof of the result for $q = p+1$. Since Cases 1, 2, and 3 prove the result for $p = 1$ and $q \geq 3$, we can assume that $p \geq 2$.

Case 4: $(q-1)l < r < ql$ and $c = 0$. Here $0 \leq (q-2)l < r-l < (q-1)l$ and the length of W is $(p+q)l$. By Equation (3), $w_{r-l} = w_r$. By our convention on the subindices,

$$x_j = v_0 = w_r = w_{r-l} = v_{-l} = v_{(p+q-1)l} = x_i.$$

contradicting the hypothesis that (i, j) is extremal.

Case 5: $(q-1)l - c < r < ql - c$ and $c > 0$, $p \geq 2$. Hence

$$l \leq (q+1)l - c - 1 - r < (q+1)l - c - 1 - (q-1)l + c = 2l - 1 < ql - c.$$

Since $c > 0$ and (k, h) is extremal, (k, h) and $(k-1, h+1)$ are equivalent under $\mathcal{R}(X, X)$. Then $x_{k-1} = \bar{x}_h$. Since $p \geq 2$, $w_{(p+q-1)l-c-1} = x_{k-1}$.

By Equation (2) (see Figure 4)

$$\bar{x}_h = x_{k-1} = w_{(q+1)l-c-1} = v_{(q+1)l-c-1-r} = v_{ql-c-1-r} = w_{ql-c-1} = x_{h-1}.$$

Then X is not cyclically reduced, contradicting our hypothesis.

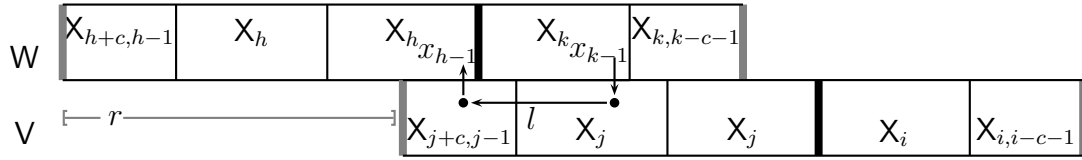


Figure 4: Case $p \geq 2$, $(q-1)l - c < r < ql - c$ and $c > 0$

Case 6: $r = ql - c$, $p \geq 2$. We have that $X_j = X_{h+c} = X_i$. Then, $x_i = x_j$ contradicting the hypothesis that (i, j) is extremal. (see Figure 5)

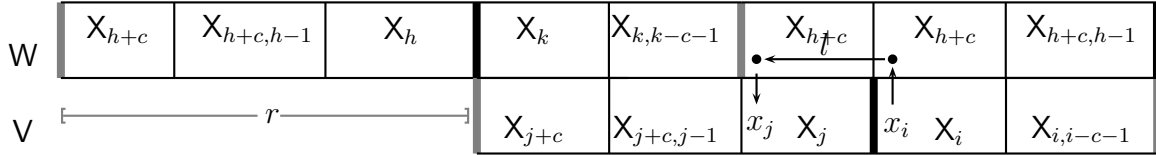


Figure 5: Case $r = ql - c$, $p \geq 2$

■

Definition 2.20. The *Manhattan norm* of an element x of \mathbb{V} denoted by $M(x)$ is the sum of the absolute values of the coefficients of the expression of x in the basis of \mathbb{V} defined by the set of cyclic words in \mathbb{A}_q . Thus if $x = c_1\mathcal{V}_1 + c_2\mathcal{V}_2 + \dots + c_n\mathcal{V}_n$ where for each $i, j \in \{1, 2, \dots, n\}$, $c_i \in \mathbb{Z}$, \mathcal{V}_i is a cyclically reduced word and $\mathcal{V}_i \neq \mathcal{V}_j$ when $i \neq j$ then $M(x) = |c_1| + |c_2| + \dots + |c_n|$. \square

The number of elements of a finite set S will be denoted by $|S|$. The following result is a consequence of Proposition 2.19 and the definition of the map $[\cdot, \cdot]$.

Proposition 2.21. *Let X be a primitive cyclically reduced linear word and let p and q be two distinct positive integers such that $p \geq 3$ or $q \geq 3$. Then, the Manhattan norm of $[\widehat{X^p}, \widehat{X^q}]$ is $p \cdot q$ times the number of elements in $\mathcal{LP}(X, X)$. In symbols,*

$$M([\widehat{X^p}, \widehat{X^q}]) = p \cdot q \cdot |\mathcal{LP}(X, X)|.$$

3 The geometric and topological side

3.1 Boundary expansions

In this Subsection we define the concept of boundary expansions and state and prove certain results of hyperbolic geometry that will be needed for Theorem 3.16. Boundary expansions were discovered by Nielsen (see [20]), who worked on the harder case of closed surfaces. Here we are constructing connected surfaces with non-empty boundary following [2] and [7].

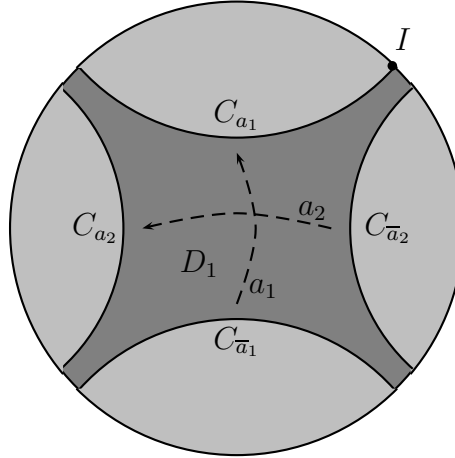


Figure 6: Example: $\mathbb{A}_q = [a_1, a_2, \bar{a}_1, \bar{a}_2]$

Let q be a positive integer and let \mathbb{A}_q denote an ordered alphabet. Denote by \mathbb{D} the open unit disk endowed with the Poincaré metric and by $\partial\mathbb{D}$ the boundary of \mathbb{D} . Endow \mathbb{D} with counterclockwise orientation. Consider $2q$ symmetrically placed disjoint geodesic arcs in \mathbb{D} with endpoints in $\partial\mathbb{D}$, such that each of them bounds disjoint half-planes. (Recall that these are arcs of circles in the usual metric on the Euclidean plane which are orthogonal to $\partial\mathbb{D}$.) Label each of these arcs using the symbols of \mathbb{A}_q as follows: Choose one of the circles and label it with C_{s_1} , where s_1 is the first symbol in \mathbb{A}_q . For each $i \in \{2, 3, \dots, 2q\}$, label the i -th circle from C_{s_1} in counterclockwise direction by C_{s_i} , where s_i is the i -th letter in \mathbb{A}_q .

Let D_1 denote the connected component of the complement of the geodesic arcs, which borders each of them.

For each $j \in \{1, 2, \dots, q\}$ consider the hyperbolic isometry defined on \mathbb{D} that maps the arc labeled with $C_{\bar{a}_j}$ to the arc labeled with C_{a_j} and whose axis is the (unique) line perpendicular to C_{a_j} and $C_{\bar{a}_j}$. Denote this isometry by a_j .

Denote by G the group generated by the isometries a_1, a_2, \dots, a_q . It is known that every non-trivial element of G is hyperbolic and that D_1 is a fundamental domain for G . (The reader is referred to [1] for definitions and proofs. In particular, see Exercise 2 at the end of Section 9.3, for tools to prove these facts). The action of G on \mathbb{D} extends naturally to $\partial\mathbb{D}$. The *limit set of G* , denoted by Λ is the smallest non-empty G -invariant closed subset of the closure of \mathbb{D} (see [1, Theorem 5.3.7 and page 188]). Observe that Λ is a Cantor set embedded in $\partial\mathbb{D}$.

Denote by \mathbb{U} the hyperbolic convex hull of Λ in \mathbb{D} ; (see [2].)

Endow \mathbb{U} with the orientation induced from \mathbb{D} . The quotient of \mathbb{U} by the action of G is a connected surface with non-empty boundary. This surface, with the orientation from \mathbb{U} , will be denoted by Σ . The quotient map $\Gamma: \mathbb{U} \longrightarrow \Sigma$ is a universal covering map for the surface Σ and the fiber of any point in Σ is in one-to-one correspondence with the group G . Therefore, the map Γ induces a bijection between G and the fundamental group of Σ . (Note that any compact connected orientable surface with non-empty boundary can be constructed in this way.)

Fix a point O in the interior of $\mathbb{U} \cap D_1$. Consider the homomorphism $\psi: G \longrightarrow \pi_1(\Sigma, \Gamma(O))$ defined for each g in G as follows: take an oriented curve C from O to $g(O)$ and set $\psi(g)$ as the (based) homotopy class of the closed curve $\Gamma(C)$. Observe that G is a free group with generators a_1, a_2, \dots, a_q .

Thus we have the following well known fact:

Proposition 3.1. *(1) The homomorphism ψ is an isomorphism and induces one to one correspondences between*

- (a) *elements of G ,*
- (b) *elements of the fundamental group of the surface Σ with basepoint $\Gamma(O)$,*
- (c) *reduced words in \mathbb{A}_q .*

(2) The above correspondence induces one to one correspondances between

- (a) *conjugacy classes of G ,*
- (b) *cyclic words on \mathbb{A}_q ,*
- (c) *free homotopy classes of curves on the surface Σ .*

We will use Proposition 3.1 to identify elements of the different sets. In particular, every cyclic word labels a unique conjugacy class.

Since all non-trivial elements of G are hyperbolic, for each element $V \in G$ there exists a unique oriented hyperbolic line on \mathbb{D} invariant by V , namely the axis of the isometry V (see [1]). We denote this line by $\text{Axis}(V)$.

Lemma 3.2. *Let V be a reduced linear primitive word. Then $\text{Axis}(V)$ covers via Γ a closed oriented curve in \widehat{V} , the conjugacy class of the element labeled by V in $\pi_1(\Sigma, \Gamma(O))$.*

By Lemma 3.2, each cyclically reduced linear word V determines a closed curve on the surface Σ , namely, $\Gamma(\text{Axis}(V))$ (which is, indeed the unique geodesic in the free homotopy class of V). We denote this curve by $\gamma(V)$.

The elements of G are composed from right to left, so if V and W are two linear words in \mathbb{A}_q then $VW(P) = V(W(P))$ for each $P \in \mathbb{U}$.

Now, we label with half-infinite words in \mathbb{A}_q , the points in the limit set Λ of G , i.e., the point in $\partial\mathbb{D}$ which are limit points for the action of G on \mathbb{D} .

Firstly we will label by $[s]$, the arc of the circumference $\partial\mathbb{D}$ that intersects the circle determined by C_s . (We maintain the convention that $\overline{s} = s$). For each reduced word $v_0v_1 \dots v_{n-1}$ define an arc of $\partial\mathbb{D}$ $[v_0v_1 \dots v_{n-1}]$ by the formula $[v_0v_1 \dots v_{n-1}] = v_0v_1 \dots v_{n-2}([v_{n-1}])$.

For a proof of the following results see [2]. As stated in [2], the proof of Lemma 3.3 is a special case of 4.9 and 4.10 in [25].

Lemma 3.3. *Let $T = t_0t_1t_2 \dots$ be a reduced half infinite word in \mathbb{A}_q .*

(1) *For each non-negative integer n , $[t_0t_1 \dots t_n] \subset [t_0t_1 \dots t_{n-1}] \subset \dots \subset [t_0]$.*

(2) $\bigcap_{n=0}^{\infty} [t_0t_1 \dots t_n] = \lim_{n \rightarrow \infty} t_0t_1 \dots t_n(O)$.

By Lemma 3.3, we can state the following definition:

Definition 3.4. Let P be a point in $\partial\mathbb{D}$ and assume that there exists a (unique) half infinite word $T = t_0t_1t_2 \dots$ such that $P = \lim_{n \rightarrow \infty} t_0t_1 \dots t_n(O)$. Then, the half infinite word T is defined to be the *boundary expansion* of P and we will refer to the point P as $t_0t_1t_2 \dots$. \square

Let $V = v_0v_1 \dots v_{n-1}$ be a finite linear word. Recall that V^∞ is the half infinite word $VV \dots$. The points with boundary expansion V^∞ and $V^{-\infty} (= \overline{V}^\infty)$ are fixed by the finite word V (see [2] for a proof). Thus, we have

Lemma 3.5. *Let $V = v_0v_1 \dots v_{n-1}$ be a cyclically reduced word, then the endpoints of $\text{Axis}(V)$ are V^∞ and $V^{-\infty}$.*

In the next lemma we study which are the fundamental domains traversed by $\text{Axis}(\mathbf{V})$, for a cyclically reduced word \mathbf{V} .

Lemma 3.6. *Let $\mathbf{V} = v_0v_1v_2 \dots v_{n-1}$ be a cyclically reduced word in \mathbb{A}_q . Then the axis of \mathbf{V} , $\text{Axis}(\mathbf{V})$, intersects D_1 . Moreover, $\text{Axis}(\mathbf{V})$ passes through the bi-infinite sequence of domains*

$$\dots \bar{v}_{n-3}\bar{v}_{n-2}\bar{v}_{n-1}(D_1), \bar{v}_{n-2}\bar{v}_{n-1}(D_1), \bar{v}_{n-1}(D_1), D_1, v_0(D_1), v_0v_1(D_1), v_0v_1v_2(D_1), \dots$$

Proof. By Lemma 3.5, the endpoints of $\text{Axis}(\mathbf{V})$ are \mathbf{V}^∞ and $\mathbf{V}^{-\infty}$. By definition, the point \mathbf{V}^∞ is in $[v_0]$ and $\mathbf{V}^{-\infty}$ is in $[\bar{v}_{n-1}]$. Since \mathbf{V} is cyclically reduced, $v_0 \neq \bar{v}_{n-1}$. Since D_1 separates the arcs $[v_0]$ and $[\bar{v}_{n-1}]$ then $\text{Axis}(\mathbf{V})$ intersects D_1 . Since \mathbf{V}^∞ is in $[v_0v_1]$, and $v_0(D_1)$ separates D_1 and $[v_0v_1]$, $\text{Axis}(\mathbf{V})$ intersects $v_0(D_1)$. The proof can be completed using the same ideas. \blacksquare

The following lemma is a direct consequence of Lemma 3.6 and the fact that, with the exception of the identity, no transformation of G fixes D_1 .

Lemma 3.7. *Let $\mathbf{V} = v_0v_1 \dots v_{n-1}$ and $\mathbf{W} = w_0w_1 \dots w_{m-1}$ be two cyclically reduced words. If $\text{Axis}(\mathbf{V})$ and $\text{Axis}(\mathbf{W})$ pass through the same translate D_2 of the fundamental domain D_1 and $D_2 \neq D_1$ then there exists a positive integer s such that the following holds*

- (1) $D_2 = v_0 \dots v_s(D_1)$ or $D_2 = \bar{v}_{n-1}\bar{v}_{n-1} \dots \bar{v}_{n-1-s}(D_1)$, and
- (2) $D_2 = w_0 \dots w_s(D_1)$ or $D_2 = \bar{w}_{m-1}\bar{w}_{m-1} \dots \bar{w}_{m-1-s}(D_1)$.

3.2 The Goldman Lie bracket

Let Σ be an oriented surface. The Goldman bracket (see [10]) is a Lie bracket defined on the free \mathbb{Z} -module generated by all free homotopy classes of oriented curves on the surface Σ . Recall that by Proposition 3.1 free homotopy classes can be identified with cyclic words in the free generators of the fundamental group of Σ . We recall the definition of the Goldman bracket: For each pair of homotopy classes (or cyclic words) \mathcal{V} and \mathcal{W} , consider representatives α and β respectively, that only intersect in transversal general position. The bracket of $\langle \mathcal{V}, \mathcal{W} \rangle$ is defined as the signed sum over all intersection points P of α and β of free homotopy class of the curve that goes around α starting and ending at P and then goes around β starting and ending at P . The sign of the term at an intersection point P is the intersection number of α and β at P .

Definition 3.8. If α and β be directed closed curves on the surface Σ . For each integer k , α^k denotes the curve that goes k times around α . By $\text{class}(\alpha)$ we denote the free homotopy class of α .

Define *transversal coincidence point* to be a pair of points (x, y) , x in the domain of α , y in the domain of β with the same image in Σ such that the branch through x is transversal to the branch through y in Σ . Denote by $\text{tcp}(\alpha, \beta)$ the set of transverse coincidence points of α and β . If (x, y) is a transversal coincidence point of α and β , denote by α_x the curve α with basepoint determined by x and by $\text{sign}(x, y)$ the sign determined by the agreement or not of the ordered branches (first that of x , then that of y) with the orientation of Σ . Also, for a transversal coincidence point (x, y) of α and β , $\alpha_x \cdot_{(x,y)} \beta_y$ denotes the based loop product of α_x and β_y . \square

Remark 3.9. The definition of the Goldman Lie bracket extends to a slightly more general circumstance than the two representatives only intersect transversally in general position (e.g., triple intersection points and power curves intersecting transversally). In the above description we sum over transverse coincidence points, the free homotopy class of α starting at x followed by β starting at y . \square

The next lemma will be used to compute bracket of powers of curves.

Lemma 3.10. *Let \mathcal{V} and \mathcal{W} be primitive non trivial free homotopy classes of curves on an orientable surface Σ . Let α and β be representatives of \mathcal{V} and \mathcal{W} respectively. Moreover, assume that α and β intersect transversally. Then*

$$\langle \mathcal{V}^k, \mathcal{W}^l \rangle = k \cdot l \sum_{(x,y) \in \text{tcp}(\alpha, \beta)} \text{sign}(x, y) \text{class}(\alpha_x^k \cdot_{(x,y)} \beta_y^l).$$

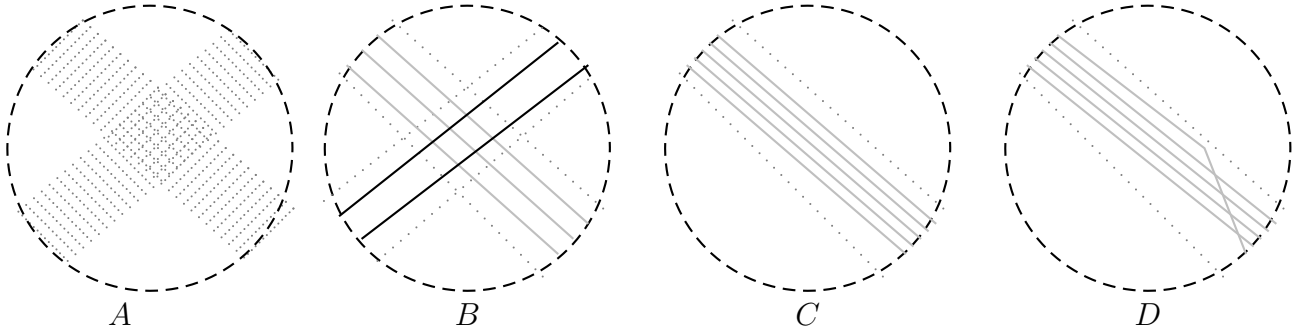


Figure 7: Proof of Lemma 3.10

Proof. First we homotope α and β to α_1 and β_1 respectively such that the intersections of α_1 and β_1 are transversal double points. Consider thin neighborhoods N_{α_1} and N_{β_1} of each of the curves, α_1 and β_1 such that these neighborhoods only intersect around the intersection points of α_1 and β_1 as in Figure 7(A). In N_{α_1} place k (resp. l) parallel copies of α_1 (resp. β_1). Thus around an intersection point of α_1 and β_1 these curves are as in Figure 7(B) and far from intersection points as in Figure 7(C).

In N_{α_1} (resp. N_{β_1}) choose a small piece of the band as in Figure 7(C) and reconnect the curves as in Figure 7(D).

Observe that around each intersection point of α_1 and β_1 there are $k.l$ intersection points of the obtained curve in N_{α_1} and the obtained curve in N_{β_1} .

Using the obtained curve in N_{α_1} and the obtained curve in N_{β_1} as representatives of α_1^k and β_1^l , and homotoping back α_1 and β_1 to α and β respectively, the result follows easily. ■

Remark 3.11. (1) The reason we do not ask in the hypothesis of Lemma 3.10 that the representatives α and β intersect in transversal *double* points is that we need to apply Lemma 3.10 to geodesic representatives, which intersect transversally but not necessarily in double points.

(2) Note that in Lemma 3.10 \mathcal{V} and \mathcal{W} can be the same free homotopy class. □

3.3 The equivalence between the combinatorial and the geometric bracket

In this section we prove that the bracket defined combinatorially in Subsection 2.4 is precisely the bracket of the Goldman Lie algebra.

The next result is Theorem A of [2].

Theorem 3.12. *Let P and Q be distinct points on $\partial\mathbb{D}$ with boundary expansions $e_1e_2\dots$ and $f_1f_2\dots$ respectively. Then P precedes Q in counterclockwise order around $\partial\mathbb{D}$ starting from the point I (see Figure 6) if and only if either*

(i) e_1 precedes f_1 in the alphabet \mathbb{A}_q or

(ii) $e_i = f_i$ for each $i \in \{1, 2, \dots, m\}$ and e_{m+1} precedes f_{m+1} in the alphabet \mathbb{A}_q with the order cyclically rotated so that \bar{e}_m is the first letter.

For every pair of cyclically reduced primitive words, denote by $i(\mathbf{V}, \mathbf{W})$ the set of transversal coincidence points of the geodesics $\gamma(\mathbf{V})$ and $\gamma(\mathbf{W})$. Observe that the cardinality of the set $i(\mathbf{V}, \mathbf{W})$ is equal to the number of intersection points of $\gamma(\mathbf{V})$ and $\gamma(\mathbf{W})$ counted with multiplicity.

Recall that $\Gamma: \mathbb{U} \longrightarrow \Sigma$ denotes the covering map. For a proof of the next result, see [7].

Proposition 3.13. ([7], page 492.) *Let V and W be two primitive cyclically reduced linear words and let $(i, j) \in \mathcal{LP}(V, W)$. Let $\phi(i, j)$ be the unique transversal coincidence point of $\gamma(V)$ and $\gamma(W)$ determined by the intersection point of $\text{Axis}(V_i)$ and $\text{Axis}(W_j)$.*

The map $\phi: \mathcal{LP}(V, W) \longrightarrow i(V, W)$ is well defined. Moreover, ϕ is a bijection.

Let \mathcal{V} and \mathcal{W} be two free homotopy classes of curves on a surface (recall Proposition 3.1). The *minimal intersection number of \mathcal{V} and \mathcal{W}* is defined as the minimal number of transversal coincidence points of pairs of representatives of \mathcal{V} and \mathcal{W} such that all intersections are transverse. The *minimal self-intersection number of \mathcal{V}* is the minimal number of self-intersection points of representatives of \mathcal{V} such that all self-intersections are transverse double points.

Remark 3.14. It is possible to give a definition of *transversal self-coincidence points*, analogous to that of transversal coincidence points. Thus, the minimal intersection number can be defined as the minimal number of transversal self-coincidence points of representatives of \mathcal{V} such that all self-intersections are transverse. \square

Geodesic representatives realize minimal intersection and self intersection numbers of conjugacy classes and the cardinality of $i(V, V)$ equals twice the minimal self-intersection number of \widehat{V} (see [9] and [7]). Thus by Proposition 3.13 we have

Proposition 3.15. *Let X be a cyclically reduced linear word on the free generators of the fundamental group of the surface Σ . Then the cardinality of $\mathcal{LP}(X, X)$ equals twice the minimal self-intersection number of X .*

We now show that, in the case of surfaces with non empty boundary, the combinatorial bracket and the Goldman bracket coincide.

Theorem 3.16. *Let X and Y be cyclically reduced linear words on the free generators of the fundamental group of the surface Σ . Then the Goldman bracket $\langle \widehat{X}, \widehat{Y} \rangle$ and the bilinear map $[\widehat{X}, \widehat{Y}]$ defined on Subsection 2.4 coincide. In symbols,*

$$\langle \widehat{X}, \widehat{Y} \rangle = [\widehat{X}, \widehat{Y}].$$

Proof. Let k and l be two positive integers and V and W two primitive cyclically reduced words such that $X = V^k$ and $Y = W^l$. By definition,

$$[\widehat{X}, \widehat{Y}] = k \cdot l \sum_{(\widetilde{i, j}) \in \mathcal{LP}(V, W)} s_{V, W(i, j)} \cdot \widehat{V_i^k W_j^l}.$$

Recall that $\gamma(V) = \Gamma(\text{Axis}(V))$ and $\gamma(W) = \Gamma(\text{Axis}(W))$. By Lemma 3.2, $\gamma(V)$ and $\gamma(W)$ are representatives of \widehat{V} and \widehat{W} respectively. Thus, by Lemma 3.10,

$$\langle \widehat{X}, \widehat{Y} \rangle = k \cdot l \sum_{(x,y) \in \text{tcp}(\mathbf{V}, \mathbf{W})} \text{sign}(x, y) \cdot \text{class}(\gamma(\mathbf{V})_{x \cdot (x,y)}^k \gamma(\mathbf{W})_y^l).$$

Let ϕ be the map of Proposition 3.13. Since ϕ is bijective, it is enough to show that for each linking pair (i, j) in $\mathcal{LP}(\mathbf{V}, \mathbf{W})$ if $\phi(\widetilde{(i, j)}) = (x, y)$ then $\mathbf{s}_{\mathbf{V}, \mathbf{W}}(i, j) = \text{sign}(x, y)$ and $\widehat{\mathbf{V}_i^k \mathbf{W}_j^l} = \text{class}(\gamma(\mathbf{V})_{x \cdot (x,y)}^k \gamma(\mathbf{W})_y^l)$.

Since the orientation on the surface Σ is induced by the orientation on \mathbb{D} , it is the same to compute $\text{sign}(x, y)$ to the sign at the intersection of $\text{Axis}(\mathbf{V}_i)$ and $\text{Axis}(\mathbf{W}_j)$ on \mathbb{D} . Thus the equality $\mathbf{s}_{\mathbf{V}, \mathbf{W}}(i, j) = \text{sign}(x, y)$ follows straightforwardly from Theorem 3.12.

We claim now that $\widehat{\mathbf{V}_i^k \mathbf{W}_j^l} = \text{class}(\gamma(\mathbf{V})_{x \cdot (x,y)}^k \gamma(\mathbf{W})_y^l)$. Indeed, let Q denote the unique point in the intersection of $\text{Axis}(\mathbf{V}_i)$ and $\text{Axis}(\mathbf{W}_j)$. Since Q is in the intersection of both axis, Q must be in a fundamental domain D_2 intersected by both axis. Assume that $D_2 \neq D_1$ (if $D_2 = D_1$ the result can be proved in a similar way). Let s be a positive integer as in Lemma 3.7 for D_2 . Assume, for instance, that

$$D_2 = v_i v_{i+1} \dots v_{i+s}(D_1) = \overline{w}_{j-1} \overline{w}_{j-2} \dots \overline{w}_{j-s-1}(D_1).$$

(the other possibilities can be treated similarly).

Then $v_i v_{i+1} \dots v_{i+s} = \overline{w}_{j-1} \overline{w}_{j-2} \dots \overline{w}_{j-s-1}$. This implies that $(i + s, j - s) \in \widetilde{(i, j)}$.

By Lemma 2.7 and Lemma 2.2(4) since $(i + s, j - s) \in \widetilde{(i, j)}$, $\widehat{\mathbf{V}_i^k \mathbf{W}_j^l} = \widehat{\mathbf{V}_{i+s}^k \mathbf{W}_{j-s}^l}$.

Consider the oriented segment S of $\text{Axis}(\mathbf{V}_i)$ from Q to $\mathbf{V}_{i+s}^k(Q)$, and the oriented segment R of $\text{Axis}(\mathbf{W}_j)$ from Q to $\mathbf{W}_{j-s}^l(Q)$. The oriented piecewise linear path obtained by following S and then $\mathbf{V}_{i+s}^k(R)$ is a lifting of the loop $\gamma(\mathbf{V})_{x \cdot (x,y)}^k \gamma(\mathbf{W})_y^l$ that starts at Q , i.e., $\Gamma(S \cdot \mathbf{V}_{i+s}^k(R)) = \gamma(\mathbf{V})_{x \cdot (x,y)}^k \gamma(\mathbf{W})_y^l$. Since $\mathbf{V}_{i+s}^k \mathbf{W}_{j-s}^l$ maps Q into $\mathbf{V}_{i+s}^k \mathbf{W}_{j-s}^l(Q)$, then the free homotopy class of $\gamma(\mathbf{V})_{x \cdot (x,y)}^k \gamma(\mathbf{W})_y^l$ is precisely $\widehat{\mathbf{V}_{i+s}^k \mathbf{W}_{j-s}^l}$. This completes the proof of the claim. \blacksquare

Remark 3.17. Turaev cobracket δ is defined on the vector space generated by the non-trivial free homotopy classes of curves. If \mathbf{V} is a non-empty cyclically reduced word then

$$\delta(\widehat{\mathbf{V}}) = \sum_{(\widetilde{i,j}) \in \mathcal{LP}(\mathbf{V}, \mathbf{V})} \mathbf{s}_{\mathbf{V}, \mathbf{V}}(i, j) \widehat{\mathbf{V}_{i,j-1}} \otimes \widehat{\mathbf{V}_{j,i-1}}$$

where $\mathbf{V}_{i,j-1} = v_i v_{i+1} \dots v_{j-1}$ and $\mathbf{V}_{i,j-1}$ does not contain \mathbf{V} as a subword (see [5].) \square

4 Applications

In Subsection 4.1 we prove Theorem 4.1 which states, roughly speaking, that for certain pair of integers p and q and for each conjugacy class \mathcal{V} , the Goldman Lie bracket $\langle \mathcal{V}^p, \mathcal{V}^q \rangle$

"computes" the minimal number of self-intersection points of \mathcal{V} . From this result, the statement that a class \mathcal{V} contains a simple representative if and only if $\langle \mathcal{V}^p, \mathcal{V}^q \rangle = 0$ (Corollary 4.2) follows directly.

Finally, in Subsection 4.2, we report of some applications of our results to the characterization of which permutations of the conjugacy classes are related to diffeomorphisms of surfaces.

4.1 Minimal intersection and self-intersection numbers of oriented curves

Recall that by Proposition 3.1 free homotopy classes of curves on the surface Σ are identified with reduced cyclic words in \mathbb{A}_q . Denote by $\pi^* = \pi^*(\Sigma)$ the set of conjugacy classes of the fundamental group of an orientable surface Σ .

For each free homotopy class \mathcal{V} , denote by $s(\mathcal{V})$ the minimal number of self-intersection of \mathcal{V} . By Proposition 2.21, Proposition 3.15 and Theorem 3.16 we obtain,

Theorem 4.1. *Let Σ be an oriented surface with boundary. Let \mathcal{V} be an element of π^* which is not a non-trivial power (i.e., \mathcal{V} is primitive). Let p and q be distinct positive integers such that $p \geq 3$ or $q \geq 3$. Then the Manhattan norm (see Definition 2.20) of the Lie bracket $\langle \mathcal{V}^p, \mathcal{V}^q \rangle$ is equal to $2 \cdot p \cdot q$ times the minimal self-intersection number of \mathcal{V} . In symbols,*

$$M(\langle \mathcal{V}^p, \mathcal{V}^q \rangle) = 2 \cdot p \cdot q \cdot s(\mathcal{V}).$$

A free homotopy class contains an embedded or simple representative if and only if its self-intersection number is zero. Thus we have.

Corollary 4.2. *Let Σ be an oriented surface and let \mathcal{V} be a free homotopy class of curves on Σ . Then \mathcal{V} contains a simple representative if and only if the Goldman bracket $\langle \mathcal{V}^p, \mathcal{V}^q \rangle = 0$ for some pair of distinct positive integers p and q such that $p \geq 3$ or $q \geq 3$.*

Remark 4.3. Observe that Theorem 4.1 holds even if the surface Σ is not connected. □

Helpful Remark 4.4. Since the Goldman Lie bracket is antisymmetric, for any cyclic word \mathcal{V} , the Goldman Lie bracket $\langle \mathcal{V}, \mathcal{V} \rangle$ is zero. Theorem 4.1 claims that for certain integers p and q the bracket $\langle \mathcal{V}^p, \mathcal{V}^q \rangle$ has exactly $2 \cdot p \cdot q$ times the minimal self intersection number of \mathcal{V} . On the other hand, the brackets $\langle \mathcal{V}^p, \mathcal{V}^q \rangle$ and $\langle \mathcal{V}, \mathcal{V} \rangle$ look so similar that at first glance one is tempted to deduce that since $\langle \mathcal{V}, \mathcal{V} \rangle = 0$ then $\langle \mathcal{V}^p, \mathcal{V}^q \rangle = 0$ for every cyclic word \mathcal{V} .

Here is an explanation of why this temptation is incorrect: Consider a geodesic representative α of \mathcal{V} and a self-intersection point P of α . The point P determines an

ordered pair of arcs of α (the order is determined by the orientation of the surface). There are $p \cdot q$ terms with positive sign of the bracket $\langle \mathcal{V}^p, \mathcal{V}^q \rangle$ determined by P and $p \cdot q$ terms of negative sign determined by P . A representative of the conjugacy class of these positive terms is the curve that goes p times around α starting at P on the direction of the first arc, followed by going q times around α on the direction of the second arc. A representative of a term of the bracket determined by P with negative sign is obtained as in the previous case but switching the order of the initial directions, that is, first α is covered p times in the direction of the second arc, and then q times in the direction of the first arc. Clearly, when $p = q$ the two terms obtained from the same pair of arcs are all equal. Thus these terms cancel. On the other hand, since this arcs are located in topologically different places of α , when $p \neq q$ the terms have a "good chance" of being different (and they are indeed different as it is proved in Theorem 4.1).

The combinatorial analog of this discussion consists in considering a linking pair (i, j) in $\mathcal{R}(\mathbf{V}, \mathbf{V})$ where \mathbf{V} is a linear representative of \mathcal{V} and noticing that (j, i) is also in $\mathcal{R}(\mathbf{V}, \mathbf{V})$. Moreover, the signs of (i, j) and (j, i) are opposite. Thus the set of integers $\{i, j\}$ plays the role of the pair of arcs of the previous discussion.

□

4.2 The mapping class group

By $\Sigma_{g,b}$ we denote an connected oriented surface with genus g and b boundary components. If Σ denotes a surface, we denote by $\text{Mod}(\Sigma)$ the *mapping class group* of Σ , that is set of homotopy classes of orientation preserving homeomorphisms of Σ .

In [6], Theorem 4.1 is applied to the curve complex $C(\Sigma)$ and using results of Korkmaz [17], Ivanov [14] and Luo [19] the following is proved.

Theorem 4.5. *Let Ω be a bijection on the set π^* of free homotopy classes of curves on an oriented surface with non-empty boundary. Suppose the following*

- (1) *If Ω is extended linearly to the free \mathbb{Z} module generated by π^* then $[\Omega(x), \Omega(y)] = \Omega([x, y])$ for all $x, y \in \pi^*$.*
- (2) *The bijection Ω commutes with "change of orientation", that is $\Omega(\bar{x}) = \overline{\Omega(x)}$ for all $x \in \pi^*$.*
- (3) *The bijection Ω commutes with "third power", that is $\Omega(x^3) = \Omega(x)^3$ for all $x \in \pi^*$.*

Then the restriction of Ω to the set of classes containing embedded curves is induced by an element of the mapping class group. Moreover, if $\Sigma \notin \{\Sigma_{1,1}, \Sigma_{2,0}, \Sigma_{0,4}\}$ then Ω is induced by a unique element of the mapping class group.

This result "supports" Ivanov's statement in [15]:

Metaconjecture *"Every object naturally associated to a surface Σ and having a sufficiently rich structure has $\text{Mod}(\Sigma)$ as its group of automorphisms. Moreover, this can be proved by a reduction theorem about the automorphisms of the curve complex $C(\Sigma)$."*

5 Computational theorems, questions and conjectures

In this section we state some conjectures and questions, as well as the evidence we found running our programs, which supports these conjectures.

5.1 Characterization of simple closed curves via the Goldman-Turaev Lie bialgebra

Corollary 4.2 characterizes the classes with simple representatives in terms of the Goldman Lie bracket. Chas [5] gave a negative answer to Turaev's question, whether the cobracket of a primitive class is zero only if the class contains a simple representative. It would be interesting, though, to use the cobracket to characterize simple classes. In this regard, we propose the following.

Conjecture 5.1. Let \mathcal{V} be a cyclic word which is not a proper power. Then \mathcal{V} contains a simple representative if and only if one of the following holds.

- (1) $\delta(\mathcal{V}^p) = 0$ for all integers p .
- (2) $\delta(\mathcal{V}^p) = 0$ for some integer $p \notin \{-1, 0, 1\}$.
- (3) $[\mathcal{V}^p, \mathcal{V}^q] = 0$ for all pairs of integers p and q .
- (4) $[\mathcal{V}^p, \mathcal{V}^q] = 0$ for any pair of distinct integers p and q such that $|p| \geq 3$ or $|q| \geq 3$.
- (5) $[\mathcal{V}, \mathcal{V}^q] = 0$ for some integer $q \in \{-2, -1, 2\}$.

□

The statements of the above conjecture concerning the cobracket, when the integer p is large enough, can be proved with the same ideas of the proof of Proposition 2.19. This is currently being written.

In a sequel of this paper, we will present a proof of the cases of statement (4) not covered by Corollary 4.2. Hence the cases of the statements that will be left open are $\{p, q\} \subset \{1, -1, 2, -2\}$ with $p \neq q$.

The algorithm to compute the bracket and cobracket presented in this paper was programmed by us. It can be found (in a user friendly form) at

<https://www.math.sunysb.edu/~moira>

The evidence we found by running the program mentioned above supports the above conjecture. More precisely we found the following.

Computational Theorem: *Let \mathcal{V} be a primitive cyclic word in a fixed alphabet \mathbb{A}_q . In Table 1, we exhibit the maximal length n of the primitive cyclic word \mathcal{V} for which the following hold.*

- (1) *In the column with header $[\mathcal{V}, \overline{\mathcal{V}}]$ there is the maximal length of \mathcal{V} (in the letters of the alphabet indicated in the header of the row) we tested for which the Manhattan norm of $[\mathcal{V}, \overline{\mathcal{V}}]$ equals two times the minimal number of self-intersection points of \mathcal{V} . In symbols, $M(\langle \mathcal{V}, \overline{\mathcal{V}} \rangle) = 2 \cdot s(\mathcal{V})$.*
- (2) *In the column with header $[\mathcal{V}, \mathcal{V}^2]$ there is the maximal length of \mathcal{V} (in the letters of the alphabet indicated in the header of the row) we tested for which the Manhattan norm of $[\mathcal{V}, \mathcal{V}^2]$ equals four times the minimal number of self-intersection points of \mathcal{V} . In symbols, $M(\langle \mathcal{V}, \mathcal{V}^2 \rangle) = 4 \cdot s(\mathcal{V})$.*
- (3) *In the column with header $\delta(\mathcal{V}^2)$ there is the maximal length of \mathcal{V} (in the letters of the alphabet indicated in the header of the row) we tested for which the number of terms of $\delta(\mathcal{V}^2)$, counted with multiplicity, equals two times the minimal number of self-intersection points of \mathcal{V} .*

Table 1: Word length tested

Alphabet	$[\mathcal{V}, \overline{\mathcal{V}}]$	$[\mathcal{V}, \mathcal{V}^2]$	$\delta(\mathcal{V}^2)$
$a, b, \overline{a}, \overline{b}$	21	14	16
$a, b, \overline{a}, \overline{b}, c, \overline{c}$	9	11	13
$a, b, \overline{a}, \overline{b}, c, d, \overline{c}, \overline{d}$	9	8	11
$a, \overline{a}, b, \overline{b}$	15	14	19

5.2 Power operations on curves

Let \mathcal{V} be an element of the set of conjugacy classes π^* . By Lemma 3.10, for every $\mathcal{W} \in \pi^*$ and for every positive integer k , there exists an element v of \mathbb{V} such that $[\mathcal{V}^k, \mathcal{W}] = k \cdot v$.

It would be interesting to explore whether the converse of the previous statement is true. More precisely,

Question 5.2. Let $\Pi: \mathbb{V} \longrightarrow \mathbb{V}$ be a linear map, such that the restriction of Π to π^* is injective and $\Pi(\pi^*) \subset \pi^*$. Moreover, assume that there exists an integer n larger than one such that for every $\mathcal{V}, \mathcal{W} \in \pi^*$ there exists $z \in \mathbb{V}$ such that $[\Pi(\mathcal{V}), \mathcal{W}] = n \cdot z$. Is it true that $\Pi(\mathcal{V}) = \mathcal{V}^n$ or $\Pi(\mathcal{V}) = \mathcal{V}^{-n}$ for every $\mathcal{V} \in \pi^*$?

5.3 Goldman Lie algebras

Our next question is about whether there is it possible to distinguish Lie algebras of curves on surfaces among all Lie algebras.

Problem 5.3. Find a characterization of a Lie algebra that is isomorphic to the Goldman Lie algebra of some surface. \square

It would be interesting to know how the Lie bialgebra recovers the surface.

Problem 5.4. Given the Goldman Lie algebra of a surface, determine the surface. \square

Let $(V, [,])$ be a Goldman Lie algebra of curves on a surface Σ . We say that an element $v \in V$ is a *geometric element* if there is a base B of V such that the following holds

- (1) b is an element of B .
- (2) There is a one to one correspondence $\Theta: B \longrightarrow \pi^*(\Sigma)$ between elements of B and free homotopy classes of curves on the surface Σ .
- (3) The map Θ when extended linearly to V , $\overline{\Theta}: V \longrightarrow \mathbb{V}$ is a Lie algebra isomorphism.

Problem 5.5. Let $(V, [,])$ be a Lie algebra of curves on a certain surface. Characterize the basis of geometric elements algebraically in terms of the bracket $[,]$ and the power operations. \square

Recall that the center of a Lie algebra $(V, [,])$ consists in the elements $v \in V$ such that $[v, w] = 0$ for every $w \in V$. In [6] it is proved that if a geometric element is in the center of the Goldman Lie algebra, then it is a peripheral curve.

Conjecture 5.6. The center of the Goldman Lie algebras of curves on an oriented surface Σ is generated by the free homotopy class of the trivial loop and the free homotopy classes of curves on Σ homotopic to a boundary component of Σ (that is, peripheral). \square

Etingof [8] proved the that the center of the Goldman Lie algebra of curves on a surface with empty boundary is generated by the class of the trivial loop.

Remark 5.7. When the surface $\Sigma_{g,b}$ has negative Euler characteristic, the maximal number of distinct isotopy classes of simple curves, non-peripheral containing pairwise disjoint representatives is $3g+b-3$ (this can be proved by dividing the surface into pairs of pants). Suppose that a Goldman Lie algebra is given, and one wants to determine the surface. If the geometric elements are also given (or if Conjecture 5.2 is true), one can apply Corollary 4.2 to find which elements of the base have simple representatives and then find the maximal number of disjoint ones by results of [10] or [6]. This number must be equal to $3g+b-3$. Now, find the elements of the geometric base of V that are in the center and those are b in number. Then the surface is determined by the genus g and the number of boundary components b . \square

Remark 5.8. Most of the above problems have a counterpart in the Turaev coalgebra. Also, it might be necessary to use the Lie bialgebra to solve the above problems. \square

References

- [1] Beardon, A. F., *The geometry of discrete groups*, GRM 91, Springer Verlag, New York, (1983).
- [2] Birman, J., Series, C., *An algorithm for simple curves on surfaces* J. London Math. Soc. (2), **29** (1984), 331-342 .
- [3] Chas, M., Sullivan, D., *String Topology*, arXiv: 9911159 [math.GT], Annals of Mathematics (to appear).
- [4] Chas, M., Sullivan, D., *Closed operators in topology leading to Lie bialgebras and higher string algebra*, arXiv: 0212358 [math.GT] The legacy of Niels Henrik Abel, (2004), 771-784.
- [5] Chas, M., *Combinatorial Lie bialgebras of curves on surfaces*, Topology **43**, (2004), 543-568. arXiv: 0105178v2 [math.GT]
- [6] Chas, M., *Minimal intersection of curves on surfaces*, arXiv:0706.2439v1 [math.GT]
- [7] Cohen, M., Lustig, M., Paths of geodesics and geometric intersection numbers I, in *Combinatorial Group Theory and Topology*, Utah, 1984, Ann. of Math. Studies **111**, Princeton Univ. Press, Princeton, 479-500, (1987).
- [8] Etingof, P., *Casimirs of the Goldman Lie algebra of a closed surface* Int. Math. Res. Not. (2006).
- [9] Freedman, M., Hass, J., Scott, P., *Closed geodesics on surfaces*, Bull. London Math. Soc. **14** , (1982), 385-391 .

- [10] Goldman, W. M., *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. Math. **85**, No.2, (1986), 263-302 .
- [11] Gonzales, A., Master Thesis, Universidad de la República, Uruguay and work in progress.
- [12] Harvey, W. J., Geometric structure of surface mapping class groups, in *Homological group theory*, Proc. Sympos., Durham, 1977, Cambridge Univ. Press, Cambridge, (1979), pp. 255–269, .
- [13] Morgan, J., Tian, G., *Ricci Flow and the Poincare Conjecture*, arXiv:math/0607607v2 [math.DG]
- [14] Ivanov, N., Automorphisms of complexes of curves and of Teichmüller spaces., in *Progress in knot theory and related topics*, Hermann, Paris, (1997), pp 113–120.
- [15] Ivanov, N., Fifteen problems about the Mapping Class group, in *Problems on Mapping Class Groups and Related Topics*, Benson Farb (editor), Providence, R.I. : American Mathematical Society, (2006), pp 71-80.
- [16] Jaco, W., *Heegaard Splittings and Splitting Homomorphisms*, Transactions of the American Mathematical Society **144**, (1969), 365-379.
- [17] Korkmaz, M., *Automorphisms of complexes of curves on punctured spheres and on punctured tori*, Topology Appl. **95**(2):, (1999), 85-111.
- [18] Luo, F., *On non-separating simple closed curves in a compact surface*, Topology **36**, No 2, (1997), 381-410.
- [19] Luo, F., *Automorphisms of the complex of curves*, Topology, **39**(2), (2000), 283–298.
- [20] Nielsen, J., *Untersuchungen zur Topologie des geschlossenen zweiseitigen Flächen*, I, Acta Math. **50**, 189-358 (1927); English transl., Investigations in the topology of closed orientable surfaces, *Jakob Nielsen: Collected Mathematical Papers, vol. 1*, (V.L. Hansen, ed.), Birkhauser, Boston, (1986), pp. 223-341.
- [21] Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, math.DG/0211159, (2002).
- [22] Perelman, G., *Finite extinction time for the solutions to the Ricci flow on certain three- manifolds*, math.DG/0307245, (2003).
- [23] Perelman, G., *Ricci flow with surgery on three-manifolds*, math.DG/0303109, (2003).
- [24] Poincaré, H., *Analysis Situs*, J. Ecole Polytech (2), **1**, (1895), 1-121 .

- [25] Series, C., *The infinite word problem and limit sets in Fuchsian groups*, Ergod. Th. and Dyn. Sys, I , (1981), 337-360.
- [26] Stallings, J. R., *How not to prove the Poincaré conjecture* Topology Seminar, Wisconsin, 1965, Ann. of Math. Studies **60**, Princeton, (1966), 83-88.
- [27] Turaev, V. G., *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. Ecole Norm. Sup. (4) **24**, No. 6, (1991), 635-704 .
- [28] Wolpert, S., *On the Symplectic Geometry of Deformations of Hyperbolic Surfaces*, Ann. Math. **117**, (1983), 207-234.

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